FUN WITH FOURIER SERIES

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Let $f \in L^1$, have period 2π, such that at each point, right- and left-hand limits exist. Then, the Fourier Series for f is

$$S(f) = \tfrac{1}{2}a_0 + \sum (a_k \cos kx + b_k \sin kx)$$

where

$$a_k = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt \cdot dt\,, \qquad b_k = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt \cdot dt$$

are the Fourier coefficients. The following is a well known result concerning these coefficients.

Riemann - Lebesgue Lemma

Both a_k and $b_k \to 0$ as $k \to \infty$.

One of the big questions in Fourier analysis is: Does $S(f) \to f$? A big answer came to us in the 1960's: (Carleson - 1966, and Hunt - 1967)

$$f \in L^p, p > 1, \;\Rightarrow\; S(f) \to f \text{ almost everywhere.}$$

We ask the following question: Does $S(f) \to f$ if we look at a specific x, for example $x = 0$? For $x = 0$, consider the partial sums of $S(f)$:

$$\begin{aligned} S_n &= \tfrac{1}{2}a_0 + \sum_{k=1}^{n} a_k \\ &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\tfrac{1}{2} + \sum_{k=1}^{n}\cos kt\right]dt \end{aligned}$$

The factor in the brackets is called the Dirichlet Kernel:

$$\begin{aligned} D_n &= \tfrac{1}{2} + \sum_{k=1}^{n}\cos kt = \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}} = \frac{\sin nt}{2\tan\frac{1}{2}} + \frac{\cos nt}{2} \\ &= \frac{\sin nt}{t} - \sin nt\left(\frac{1}{t} - \frac{1}{2\tan\frac{1}{2}}\right) + \tfrac{1}{2}\cos nt \end{aligned}$$

Hence, by the Riemann-Lebesgue Lemma, for any $\delta > 0$,

$$S_n = \tfrac{1}{\pi}\int_{-\delta}^{\delta} f(t)\frac{\sin nt}{t}dt + o(1)$$

An analysis of the last integral gives the following well known result.

Dirichlet - Jordan Theorem

$$f \in BV[0, 2\pi] \Rightarrow S_n \to \tfrac{1}{2}[f(0+) + f(0-)],$$

where BV[0, 2π] is the space of functions of bounded variation on [0, 2π].

(See [Z] for a thorough discussion on the above preliminaries.)

In 1972, Waterman extended this result:

Waterman (1972) [W72]

$$f \in HBV[0, 2\pi] \Rightarrow S_n \to \tfrac{1}{2}[f(0+) + f(0-)],$$

where $f \in$ HBV is defined as follows:

There exists a number M so that for any collection of non-overlapping subintervals, $\{I_k = [s_k, t_k]\}$

$$\sum \frac{|f(t_k) - f(s_k)|}{k} < M.$$

Our work deals with another question: What if we apply a summability method to the Fourier Series of f, for example Summability by the First Cesaro Mean or (C, 1) Summability, which is defined as follows:

Let $\{x_n\}$ be a sequence. We say that $\{x_n\}$ is **(C, 1) summable to x** if

$$\frac{1}{m}\sum_{n=1}^{m} x_n \to x \text{ as } m \to \infty.$$

For example, $\{(-1)^n\}$ is (C, 1) summable to 0.

Above, we saw that the partial sums of the Fourier series of f at $x = 0$ are

$$S_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\tfrac{1}{2} + \sum_{k=1}^{n} \cos kt\right]dt.$$

If we apply the (C,1) summability method, we must consider

$$\sigma_m = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{m+1}\sum_{n=0}^{m}\left(\tfrac{1}{2} + \sum_{k=1}^{n} \cos kt\right)\right]dt$$

A classical theorem on the subject from Fejér follows. (Again, see [Z].)

Fejér's Theorem

$$\sigma_m \to \tfrac{1}{2}[f(0+) + f(0-)].$$

In 1995, Schembari and Waterman [SW] took this one step further. Since the Fourier series of f is defined by

$$\tfrac{1}{2}a_0 + \sum (a_k \cos kx + b_k \sin kx),$$

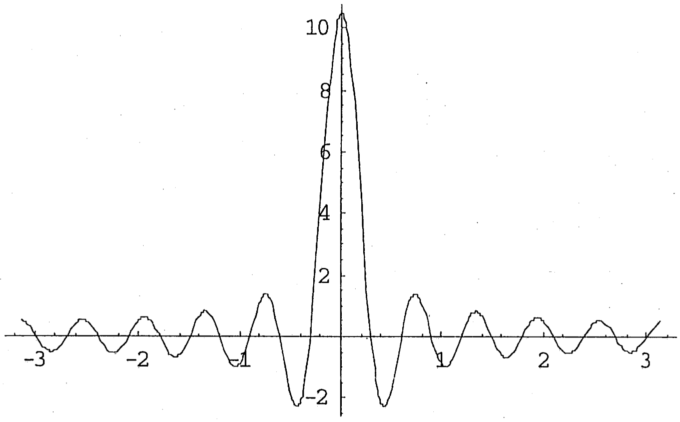
we can "formally" differentiate it to obtain

$$\text{DFS} = \sum (kb_k \cos kx - ka_k \sin kx)$$

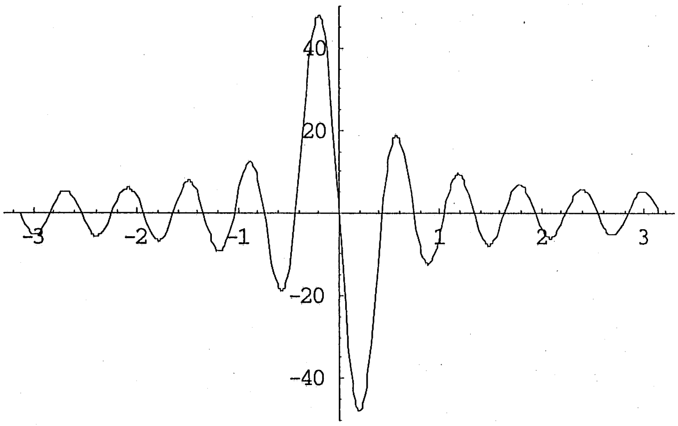
At $x = 0$, the partial sums of the DFS are:

$$S_n' = \sum_{k=1}^{n} kb_k = -\tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t) D_n'(t) \cdot dt.$$

The derivative of the Dirichlet kernel behaves quite wildly. For example, here is a graph of the 10th Dirichlet kernel:

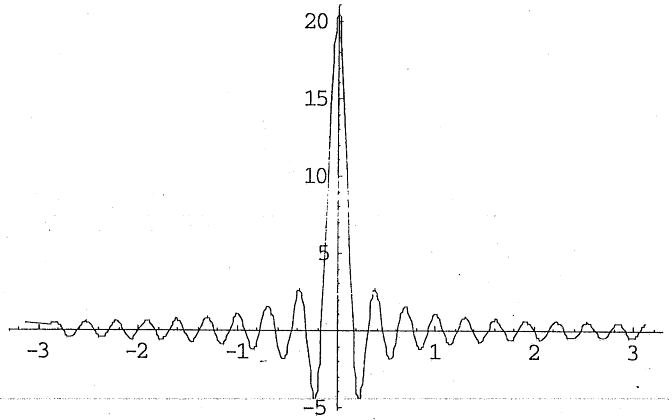


while here is a graph of the derivative of the 10th Dirichlet kernel:

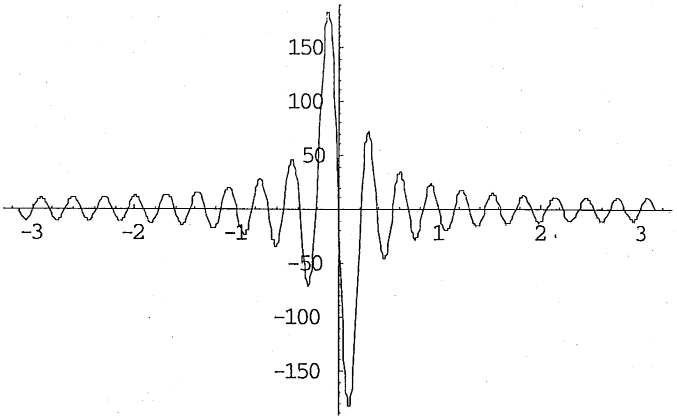


We see a similar difference by looking at the 20th kernels.

The 20th Dirichlet kernel:



The derivative of the 20th Dirichlet kernel:



Hence a comparison of the 20th kernels shows that the derivative is an order of magnitude larger in absolute value than the original kernel.

Here is even more evidence of the wild behavior of the derivative of the Dirichlet kernel. If f is absolutely continuous on $[-\pi, \pi]$, then we can apply integration by parts to obtain

$$S_n' = \sum_{k=1}^{n} kb_k = -\tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n'(t)\cdot dt = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f'(t)D_n(t)\cdot dt\,,$$

and by Waterman's theorem, we need $f' \in \mathrm{HBV}$ to insure convergence of the last integral.

We can attempt to calm down DFS by using (C, 1) Summability. We have

$$S_n' = \sum_{k=1}^{n} kb_k = -\tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n'(t)\cdot dt$$

Hence, the (C, 1) means of these partial sums are:

$$\sigma_m{}' = -\tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{1}{m+1}\sum_{n=0}^{m}\left(D_n{}'(t)\right)dt = -\tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cdot K_m{}'(t)\cdot dt$$

where K is the Fejér kernel:

$$\begin{aligned} K_m(t) &= \frac{1}{m+1}\sum_{n=0}^{m}\left(\tfrac{1}{2}+\sum_{k=1}^{n}\cos kt\right) = \frac{1}{m+1}\sum_{n=0}^{m}\frac{\sin(n+{}^{1}\!/_{2})t}{2\sin {}^{t}\!/_{2}} \\ &= \frac{1}{m+1}\cdot\frac{1-\cos(m+1)t}{(2\sin {}^{t}\!/_{2})^2} = \frac{1}{2(m+1)}\left(\frac{\sin(m+1)\,{}^{t}\!/_{2}}{\sin {}^{t}\!/_{2}}\right)^2 \end{aligned}$$

Hence,

$$\begin{aligned} K_m{}'(t) &= \frac{1}{2(m+1)}\left(\frac{\sin\frac{(m+1)t}{2}}{\sin {}^{t}\!/_{2}}\right)\left(\frac{(m+1)\cos\frac{(m+1)t}{2}\cdot\sin {}^{t}\!/_{2} - \sin\frac{(m+1)t}{2}\cdot\cos {}^{t}\!/_{2}}{\sin^2 {}^{t}\!/_{2}}\right) \\ &= \frac{\sin(m+1)t}{4\sin^2 {}^{t}\!/_{2}} - \frac{K_m(t)}{\tan {}^{t}\!/_{2}} \end{aligned}$$

With analysis similar to that for D_n, for any fixed $\delta > 0$, we can obtain

$$\sigma_m{}' = -\tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cdot K_m{}'(t)\cdot dt = -\tfrac{1}{\pi}\int_{-\delta}^{\delta} f(t)\frac{\sin mt}{t^2}dt + o(1)\,.$$

<u>In summary,</u>

The Fourier Series of f at $x = 0$ is

$$S_n = \tfrac{1}{2}a_0 + \sum a_k = = \tfrac{1}{\pi}\int_{-\delta}^{\delta} f(t)\frac{\sin nt}{t}dt \; + o(1)$$

and we have the classic theorem updated by Waterman:

Dirichlet - Jordan - Waterman

If $f \in$ BV or HBV, then $S_n \to$ ½[$f(0+) + f(0-)$].

Now, the DFS of f at $x = 0$ is

$$S_n{}' = \sum_{k=1}^{n} kb_k = -\tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n{}'(t)\cdot dt\,,$$

but $D_n{}'$ are very wild so take averages by using (C,1) means:

$$\begin{aligned}\sigma_m{}' &= \frac{1}{m+1}\sum_{n=0}^{m} S_n{}' = -\tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t) K_m{}'(t)\cdot dt \\ &= -\tfrac{1}{\pi}\int_{-\delta}^{\delta} f(t)\frac{\sin mt}{t^2}dt + o(1)\end{aligned}$$

We ask the question, "Can we extend the Dirichlet - Jordan - Waterman theorem to these means?"

The first extension is quite trivial:

Extension 1:
If $f \in$ AC, $f'(0+)$ and $f'(0-)$ exist, then the DFS is (C, 1) summable to ½[$f'(0+) + f'(0-)$].

Proof:

$$\sigma_m{}' = -\tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)K_m{}'(t)\cdot dt = -\tfrac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\cdot K_m(t)\cdot dt\,,$$

and apply Fejer's Theorem. •

The second extension follows from the Dirichlet - Jordan - Waterman theorem, and was reported in [SW] using Waterman's method in [W77].

Extension 2:
If $\dfrac{f(t)}{t} \in$ HBV, $f'(0+)$ and $f'(0-)$ exist, then the DFS is (C, 1) summable to ½[$f'(0+) + f'(0-)$].

Proof: $\sigma_m{}' = -\tfrac{1}{\pi}\int_{-\delta}^{\delta} f(t)\dfrac{\sin mt}{t^2}dt + o(1) = -\tfrac{1}{\pi}\int_{-\delta}^{\delta} \dfrac{f(t)}{t}\dfrac{\sin mt}{t}dt + o(1),$

and apply Waterman's Theorem. •

The final extension would be nice if it could be done:

Extension 3?
Can we define a generalized AC class which will give (C, 1) summability of the DFS?

Consider the following definitions:

Bounded Variation (BV)

f is of bounded variation if there exists a number M so that for any collection of non-overlapping subintervals, $\{I_k = [s_k, t_k]\}$

$$\sum \left| f(t_k) - f(s_k) \right| < M .$$

Harmonic Bounded Variation (HBV)

f is of harmonic bounded variation if there exists a number M so that for any collection of non-overlapping subintervals, $\{I_k = [s_k, t_k]\}$

$$\sum \frac{\left| f(t_k) - f(s_k) \right|}{k} < M .$$

Absolutely Continuous (AC)

f is absolutely continuous if given any number $\varepsilon > 0$, there exists a number $\delta > 0$ so that for any collection of non-overlapping subintervals, $\{I_k = [s_k, t_k]\}$

$$\sum (t_k - s_k) < \delta \Rightarrow \sum \left| f(t_k) - f(s_k) \right| < \varepsilon .$$

Harmonic Absolutely Continuous (HAC)

f is harmonic absolutely continuous if given any number $\varepsilon > 0$, there exists a number $\delta > 0$ so that for any collection of non-overlapping subintervals, $\{I_k = [s_k, t_k]\}$

$$\sum \frac{t_k - s_k}{k} < \delta \Rightarrow \sum \frac{\left| f(t_k) - f(s_k) \right|}{k} < \varepsilon .$$

An open question is: Does HAC give (C,1) summability of the DFS?

HAC does have some nice properties that lead us to hope that we can answer this question in the affirmative:

- $AC \subset HAC \subset BVC \subset HBVC$, where BVC is the space of continuous functions of bounded variation and HBVC is the space of continuous functions of harmonic bounded variation.
- If f is in HBV, then the Fourier coefficients a_n and b_n are $O\left(\frac{1}{\sum_{k=1}^{n} \frac{1}{k}} \right)$.
- If f is in HAC, then a_n and b_n are $o\left(\frac{1}{\sum_{k=1}^{n} \frac{1}{k}} \right)$.

If HAC does give (C, 1) summability of the DFS, it may lead to an interesting result about integration:

Do we have a new class of functions where we can apply integration by parts?

If $f \in$ HAC

$$\sigma_m' = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)K_m'(t)\cdot dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\cdot K_m(t)\cdot dt\ ?$$

The second integral converges by Fejér's theorem, but does the second integral equal the first?

As we see, analyzing some of the classical theorems in Fourier Analysis and attempting to apply them to other trigonometric series leads us to *Fun with Fourier Series*.

References

[SW] N. P. Schembari and D. Waterman, (C,1) summability of the differentiated Fourier series, *Journal of Mathematical Analysis and Applications* **191** (1995), pp. 633 - 646.

[W72] D. Waterman, On convergence of Fourier series of functions of generalized bounded variation, *Studia Math* **44** (1972), pp. 107 - 117; Errata *Studia Math* **44** (1972), p. 651.

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[Z] A. Zygmund, "Trigonometric Series," Volumes I, II; Cambridge University Press, Cambridge, 1988.