Building the Toughest Networks

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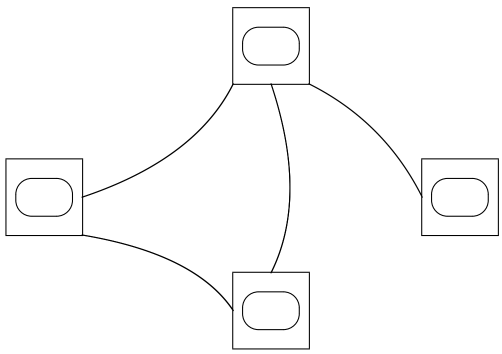
Abstract

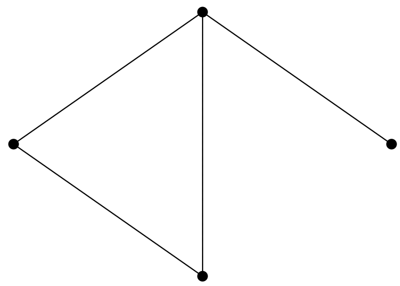
Given n computers and m cables with which to make direct connections, what is the best network that can be formed? Of course, it depends upon how best is defined. We use the graph parameter toughness as our measure, and discuss what is known for various values of n and m. We will also discuss the rough frontier of values of n and m for which answers are not known.

1 Introduction

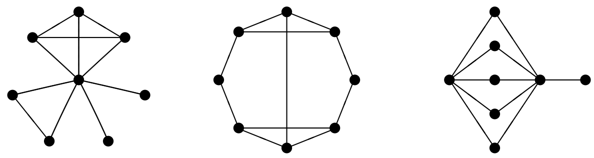
We represent a computer network by a **graph**. Computers are represented by points •. A direct cable connection between two computers is represented by a line (or curve) ———— joining the corresponding pairs of points. The points are the **vertices** and the lines are the **edges** of the graph. For example, the following computer network



is represented by the following graph.



Question 1.1. *If we have* 8 *computers and* 11 *cables, then what network is best?*



How do we measure the "goodness" of a network (graph)? We want to minimize vulnerability to computer failures or sabotage. We want the saboteur to get the least possible "bang for his buck".

(i) A **disconnecting set** for a graph is some subset of the *vertices* whose removal leaves the resulting graph in more than one piece.

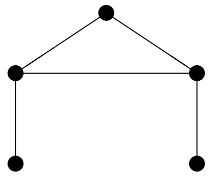
(ii) The **toughness** [1] of a graph G, denoted $\tau(G)$, is the minimum possible fraction $\frac{D}{P}$ such that D is the size of a disconnecting set and P is the number of pieces into which G is broken by the removal of S.

From the saboteur's point of view, $\frac{D}{P} = \frac{\text{buck}}{\text{bang}}$ He cares about the smallest such fraction.

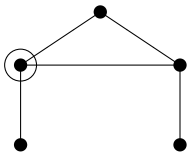
$$\tau(G) = \min\{\frac{|S|}{\omega(G-S)} : S \text{ disconnects } G\}$$

A disconnecting set S such that $\tau(G) = \frac{|S|}{\omega(G-S)} = \frac{D}{P}$ is called a τ**-set** for G. A saboteur will destroy a τ-set to be most efficient in his sabotage.

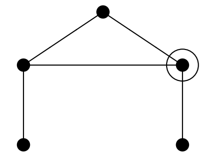
Example 1.2. What is the toughness of the pictured graph G?



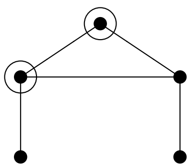
It is the smallest fraction from the set below.



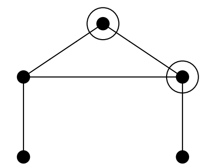
$\frac{1}{2}$



$\frac{1}{2}$



$\frac{2}{2} = 1$



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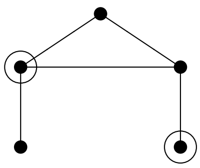
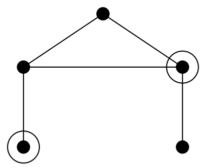
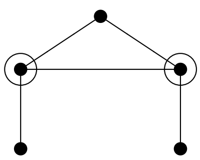


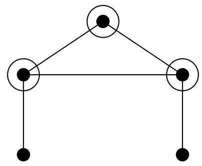
$\frac{2}{2} = 1$



$\frac{2}{2} = 1$



$\frac{2}{3}$



$\frac{3}{2}$

So $\tau(G) = \frac{1}{2}$.

From *our* point of view, if toughness is low, then the graph is weak (vulnerable). If toughness is high, then the graph is strong (tough).

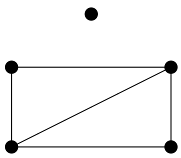
2 General Problem

If we have a fixed number n of computers and a fixed number m of cables with which to connect them in a network, then we want our network to be as strong as possible. We want to maximize toughness subject to n and m.

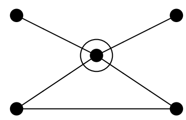
Given n and m, the **maximum toughness** [1, 2, 3, 4], denoted $T_n(m)$, is the largest of all toughness values for graphs on n vertices and m edges.

$$\begin{aligned} T_n(m) &= \max_{(n,m)-\text{graphs } G} \{\tau(G)\} \\ &= \max_{(n,m)-\text{graphs } G} \{ \min_{\text{disconn. sets } S} \{ \frac{|S|}{\omega(G-S)} \} \} \end{aligned}$$

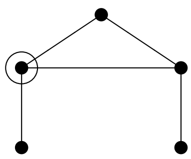
Example 2.1. What is the value of $T_5(5)$?



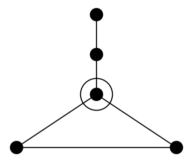
$\tau = 0$



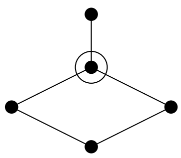
$\tau = \frac{1}{3}$



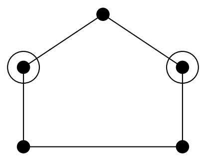
$\tau = \frac{1}{2}$



$\tau = \frac{1}{2}$



$\tau = \frac{1}{2}$

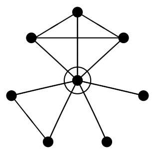


$\tau = \frac{2}{2} = 1$

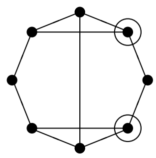
Answer: $T_5(5) = 1$.

In general, $T_n(n) = 1$ (use an n-cycle).

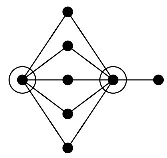
Example 2.2. What is the value of $T_8(11)$?



$\tau = \frac{1}{4}$

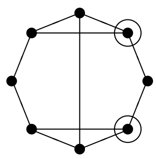


$\tau = \frac{2}{2} = 1$



$\tau = \frac{2}{6} = \frac{1}{3}$

There are 980 different graphs with 8 vertices and 11 edges. Of them, 814 are connected. In fact, $T_8(11) = 1$.



$\tau = \frac{2}{2} = 1$

Moreover,

$$1 = T_8(8) = T_8(9) = T_8(10) = T_8(11) = 1.$$

However, $T_8(12) = \frac{4}{3}$.

3 Techniques and Results

We need general theorems to compute maximum toughness values.

$$\tau(G) \leq \frac{\text{minimum degree of } G}{2}$$

If G has n vertices and m edges, then

$$\text{minimum degree of } G \leq \lfloor \frac{2m}{n} \rfloor.$$

So,

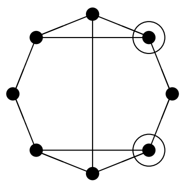
$$T_n(m) \leq \frac{\lfloor \frac{2m}{n} \rfloor}{2}.$$

Example 3.1. We compute $T_8(11)$ in two steps.

(i) For $n = 8$ and $m = 11$, we have

$$T_8(11) \leq \frac{\lfloor \frac{2m}{n} \rfloor}{2} = \frac{\lfloor \frac{2(11)}{8} \rfloor}{2} = \frac{\lfloor 2.75 \rfloor}{2} = \frac{2}{2} = 1.$$

(ii) The pictured graph has toughness 1.

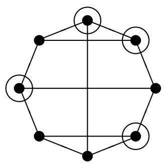


$\tau = \frac{2}{2} = 1$

Therefore, $T_8(11) = 1$.

For any fixed value of n, potential changes in $T_n(m)$ occur when $\lfloor \frac{2m}{n} \rfloor$ increments.

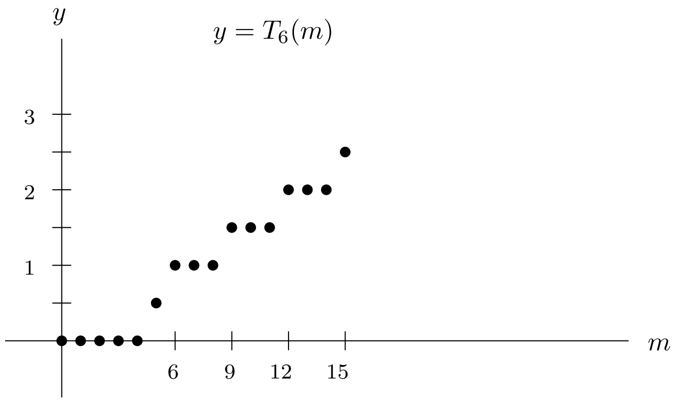
Example 3.2. If $n = 8, m = 11$, then $\lfloor \frac{2m}{n} \rfloor = 2$.
If $n = 8, m = 12$, then $\lfloor \frac{2m}{n} \rfloor = 3$.
It might seem possible that $T_8(12) = \frac{3}{2}$. However, $T_8(12) = \frac{4}{3}$.



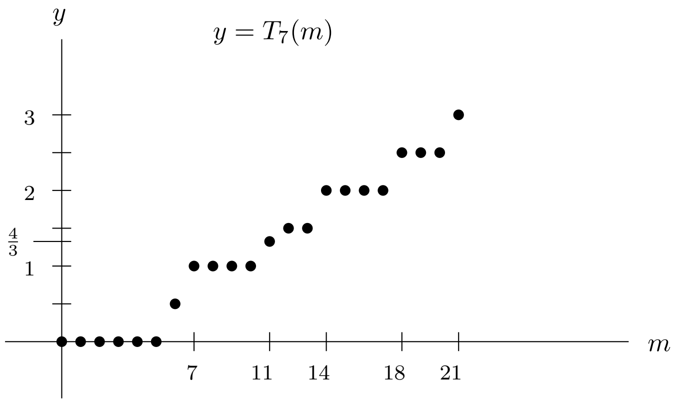
$\tau = \frac{4}{3}$

$m \rightarrow$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$n=3$	0	0	$\frac{1}{2}$	1												
$n=4$	0	0	0	$\frac{1}{2}$	1	1	$\frac{3}{2}$									
$n=5$	0	0	0	0	$\frac{1}{2}$	1	1	1	$\frac{3}{2}$	$\frac{3}{2}$	2					
$n=6$	0	0	0	0	0	$\frac{1}{2}$	1	1	1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	2	2	2	$\frac{5}{2}$

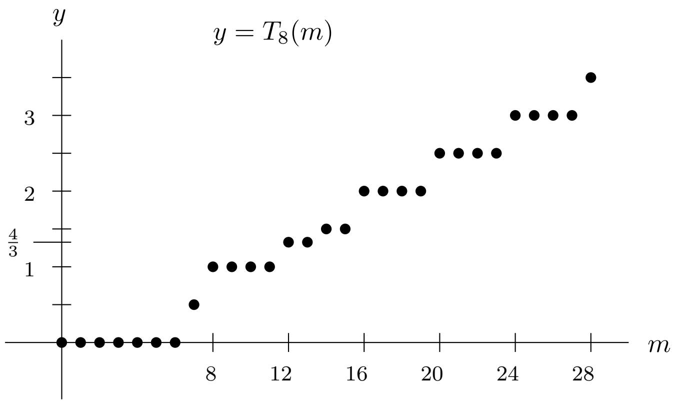
Table 1: Values of $T_n(m)$ for Small n



y
y = T6(m)
3
2
1
m
6
9
12
15



y
y = T7(m)
3
2
4/3
1
m
7
11
14
18
21



y
y = T8(m)
3
2
4/3
1
m
8
12
16
20
24
28

For each r, T_n 'flattens out' over the interval

$$\underbrace{\left\lceil \frac{nr}{2} \right\rceil} \leq m < \left\lceil \frac{n(r+1)}{2} \right\rceil$$

in which $\delta(G) = r$ is possible. We consider degree sequences $r, r, \ldots, r$ or $r+1, r, \ldots, r$, i.e. nearly regular graphs.

In general $T_n(m) \leq \frac{r}{2}$ where $r = \lfloor \frac{2m}{n} \rfloor$. When is equality possible? Equivalently, when is $T_n\left(\lceil \frac{nr}{2} \rceil\right) = \frac{r}{2}$ possible?

Equality is achieved when r is even by Harary graphs [5]. When r is

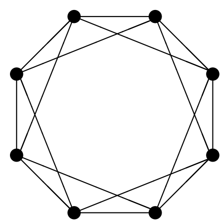


Table 2: The Harary graph $H(8, 16)$

odd, then question is harder.

Theorem 3.3 ($r = 3$, 2005)**.** *Let* $n \geq 5$*.*

(a) If $n \equiv 0$ *or* $5 \mod 6$*, then*

$$T_n(m) = \frac{3}{2} \quad \textit{for } \lceil \tfrac{3n}{2} \rceil \leq m < 2n.$$

(b) If $n \equiv 1, 3$ *or* $4 \mod 6$*, then*

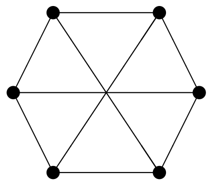
$$T_n(m) = \begin{cases} \frac{3\lfloor \frac{n}{6} \rfloor + 1}{2\lfloor \frac{n}{6} \rfloor + 1} & \textit{for } m = \lceil \frac{3n}{2} \rceil, \\ \frac{3}{2} & \textit{for } \lceil \frac{3n}{2} \rceil + 1 \leq m < 2n. \end{cases}$$

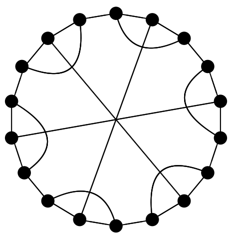
(c) If $n \equiv 2 \mod 6$*, then*

$$T_n(m) = \begin{cases} \frac{3\lfloor \frac{n}{6} \rfloor + 1}{2\lfloor \frac{n}{6} \rfloor + 1} & \textit{for } m = \lceil \frac{3n}{2} \rceil, \\ \frac{3}{2} & \textit{for } \lceil \frac{3n}{2} \rceil + 2 \leq m < 2n. \end{cases}$$

and $\frac{3\lfloor \frac{n}{6} \rfloor + 1}{2\lfloor \frac{n}{6} \rfloor + 1} \leq T_n(\lceil \frac{3n}{2} \rceil + 1) \leq \frac{3}{2}$*.*
Moreover, $T_8(13) = \frac{4}{3}$*.*

We consider inflations of Cubic Harary Graphs.





Theorem 3.4 ($r = 5$, 2006)**.**
For $n \geq 6$ with $n \notin \{11, 17, 18, 19, 21, 33\}$,

$$T_n(\lceil \tfrac{5n}{2} \rceil) = \tfrac{5}{2}.$$

When $r = 5$, we construct a family of $\frac{5}{2}$-tough $(n, \lceil \frac{5n}{2} \rceil)$-graphs $G_5(n)$ that, moreover, contain multiple $K_{1,3}$-centers. What about odd r in general? $r = 7$? $r \geq 9$? These harder questions are the subject of ongoing work. What about large odd r? Harary graphs work for that too.

Theorem 3.5 (2002)**.** *Let $n \geq 3$. If*

- *(i) r is even, or*
- *(ii) $r \geq 2\lfloor \frac{n}{3} \rfloor$,* *(i.e. $m \geq n\lfloor \frac{n}{3} \rfloor$)*

then $T_n(\lceil \frac{nr}{2} \rceil) = \frac{r}{2}$.

Theorem 3.6 (2006)**.** *Let n be even and $r \geq \frac{n}{2}$. Then, $T_n(\frac{nr}{2}) = \frac{r}{2}$.*

Theorem 3.7 (2006)**.** *Let n be odd and $r = \frac{n + (n \bmod 4)}{2}$.*

- *(a) If $n \equiv 1 \pmod 4$, then*
  $T_n(\lceil \frac{nr}{2} \rceil + 1) = \frac{r}{2}$.
- *(b) If $n \equiv 3 \pmod 4$ and $n \geq 19$, then*
  $T_n(\lceil \frac{nr}{2} \rceil + 1) \geq \frac{r}{2} - \frac{1}{6}$.

References

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