

André and the Ballot Problem - History and a Generalization

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Abstract We describe the ballot problem, give a well-known proof utilizing the “reflection method,” and present the first combinatorial proof as given by Désiré André (the reflection method is typically misattributed to André). We present the generalized ballot problem and see that the reflection method fails to provide a proof, whereas André’s original proof can be modified to handle the general case. Much of the material in this paper summarizes the author’s previous work [10].

1. THE BALLOT PROBLEM. The ballot problem is the following:

Suppose that two candidates, A and B , are in an election where candidate A receives a votes, candidate B receives b votes, and $a > b$. How many ways can the $a + b$ ballots be ordered so that *while the ballots are being counted*, candidate A maintains a constant lead over B ?

For example, suppose the ballots are marked “A” and “B”, and A gets 5 votes and B gets 3 votes. Then we wish to count ballot permutations such as

A A B A B A A B

but not

A A B A B B A A.

This problem was originally posed by Joseph Bertrand in 1887 [5]. (He actually asked for the *probability* that A stays ahead of B , but most sources now state the problem as above.) Bertrand also supplied the answer: $\frac{a-b}{a+b} \binom{a+b}{a}$. He sketched an inductive proof and asked if a “direct” proof could be found.

That same year, Désiré André provided a short combinatorial proof of the solution [3], which was celebrated among mathematicians of the time; Bertrand himself included André’s proof in a probability book he published

in 1888 [6]. André’s key idea was to count the number of “bad” ballot permutations (those where A and B tie at some point) and subtract that from the number of all ballot permutations, $\binom{a+b}{a}$, to obtain the number of “good” ballot permutations (those in which A leads throughout the counting of the ballots).

In the ensuing years, mathematicians produced many variations of André’s proof. See, for example, [1, 8, 9, 13], and see also Aeppli’s proof [2] as it appears in Takács’ article [12].

The most elegant of these variations on André’s proof is a proof by the “reflection method” which appears to have its genesis in the pair of 1923 papers by Aebly [1] and Mirimanoff [9]. Proof by the reflection method is routinely misattributed to André; see the website in reference [10] for a list of sources that link André with the reflection method. In the next section we present the reflection method. André’s original proof seems to have been forgotten by recent mathematicians (an exception occurs in [12]) and we discuss that proof in section 3.

2. THE REFLECTION METHOD. We can express ballot permutations as lattice paths in the Euclidean plane by thinking of votes for A as upsteps $(1, 1)$ and votes for B as downsteps $(1, -1)$. Ballot permutations (or paths) that satisfy the ballot problem are called “good,” while those that do not are called “bad.” Feller [7, ch. III] gives a nice proof using the reflection method, which we follow here.

The ballot problem can be given the following graphical interpretation.

Given positive integers a, b with $a > b$, find the number of lattice paths starting at the origin and consisting of a upsteps $(1, 1)$ and b downsteps $(1, -1)$ such that no step ends on the x -axis.

Solution by Reflection: Let T denote the terminal point of the path, $(a + b, a - b)$. Since every good path must start with an upstep, there are as many good paths as there are paths from $(1, 1)$ to T that never touch the x -axis. The set of paths from $(1, 1)$ to T that *do* touch the x -axis somewhere is in one-to-one correspondence with the set of all paths from $(1, -1)$ to T ; this is seen by reflecting across the x -axis the initial segment of the path that ends with the step that first touches the x -axis. Subtracting the number of these paths from the number of all paths from $(1, 1)$ to T produces the number of good paths:

$$\binom{a+b-1}{a-1} - \binom{a+b-1}{b-1} = \frac{a-b}{a+b} \binom{a+b}{a}. \quad \square$$

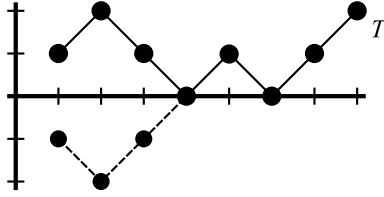


Figure 1: A ballot permutation represented as a lattice path

3. ANDRÉ'S METHOD. As mentioned previously, André's approach is to count the number of bad ballot permutations and subtract that from the number of all ballot permutations to obtain the number of good ballot permutations.

First, André observes that every ballot permutation starting with a B must be bad, and there are $\binom{a+b-1}{b-1}$ of these.

Next, André counts the number of bad ballot permutations starting with A. *Claim:* these number the same as all ballot permutations consisting of a A's and $b-1$ B's, namely $\binom{a+b-1}{b-1}$. André proves the claim with the following bijection.

(\rightarrow) Consider a bad ballot permutation starting with A. Reading from left to right, find the first B that causes the number of A's to equal the number of B's. Remove that B, causing the permutation to split into two parts. Interchange the parts and join them together to create a permutation with a A's and $b-1$ B's.

(\leftarrow) Consider any ballot permutation with a A's and $b-1$ B's. Reading from right to left, find the first A that causes the tail of the permutation (starting with that A) to contain one more A than B; this must occur since $a > b-1$. Take off the tail of the permutation, put it in front, and insert a B between the two parts. This creates a bad ballot permutation starting with A. This finishes the proof of the claim.

Thus, the number of good ballot permutations is

$$\binom{a+b}{a} - 2\binom{a+b-1}{b-1} = \frac{a-b}{a+b}\binom{a+b}{a} \quad \square$$

Example

(\rightarrow) Given a bad ballot permutation starting with A:

A A B A B B A A

Find the first "offending B."

A A B A B B A A

Remove it.

A A B A B A A

Interchange the two parts.

A A A A B A B

This is a ballot permutation with one less B.

(←) Given any ballot permutation with a A's and $b - 1$ B's where $a > b - 1$:

A A A A B A B

Identify the smallest "tail" of the permutation containing one more A than B.

A A A A B A B

Remove the tail, place at the beginning of the permutation.

A A B A B A A

Insert a B between the two parts.

A A B A B B A A

The result is a bad ballot permutation with a A's, b B's, and it starts with A.

Reflecting a lattice path across the x -axis is equivalent to taking a ballot permutation and changing the A's into B's and the B's into A's. André's proof, by contrast, uses no geometry, and instead of modifying a portion of a ballot permutation, he interchanges two parts of it. It is safe to say that André had no reflection in mind when he constructed his proof.

4. GENERALIZING THE BALLOT PROBLEM. After Bertrand posed the ballot problem and even before André submitted a direct proof, Émile Barbier provided a generalization of the ballot problem and its solution [4]. Although Barbier's actual statement is somewhat unclear, it can loosely be interpreted to state the "generalized ballot problem".

Suppose that two candidates, A and B , are in an election where candidate A receives a votes, candidate B receives b votes, and $a > kb$ for some positive integer k . How many ways can the $a + b$ ballots be ordered so that throughout the counting of the ballots, candidate A maintains *more than k times as many votes as B* ?

Barbier gives the solution as $\frac{a-kb}{a+b} \binom{a+b}{a}$, though he offers no proof. Presumably he proved it with induction as Barbier had done for the case $k = 1$.

For example, suppose A gets 7 votes and B gets 3 votes. We may consider $k = 2$, in which case we wish to count ballot permutations such as

A A A B A A B A A B

but not

A A A B A B A A A B.

We could also consider the case $k = 1$ in which we would count both of the above ballot permutations. In the case $k = 2$ there are $\frac{1}{10} \binom{10}{3} = 24$ such ballot permutations, and in the case $k = 1$ there are $\frac{4}{10} \binom{10}{3} = 96$ such ballot permutations.

It is an interesting fact that the reflection method, for all its elegance, cannot be modified to handle the generalized ballot problem. Even more interesting is the fact that André's method *can* be modified to prove the generalized ballot problem.

Sketch of proof: We consider the following graphical interpretation of the generalized ballot problem.

Given positive integers a, b, k with $a > kb$, find the number of lattice paths starting at the origin and consisting of a upsteps $(1, 1)$ and b downsteps $(1, -k)$ such that no step ends on or below the x -axis.

We call a downstep “bad” if it starts above the x -axis and ends on or below the x -axis. For $0 \leq i \leq k$ let \mathcal{B}_i denote the set of bad paths whose first bad downstep ends i units below the x -axis (observe that the paths in \mathcal{B}_k are exactly those that start with a downstep). Let \mathcal{A} be the set of all paths consisting of a upsteps and $b - 1$ downsteps, without regard to location in the plane; $|\mathcal{A}| = \binom{a+b-1}{a}$. See Figure 2.

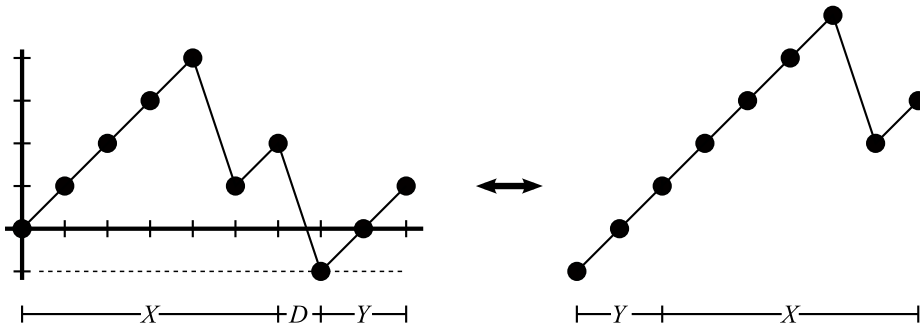


Figure 2: Example with $k = 3$. $XDY \in \mathcal{B}_1$ and $YX \in \mathcal{A}$.

One can show that $|\mathcal{B}_i| = |\mathcal{A}|$ for each i in the range $0 \leq i \leq k$. This can be done using André's method of removing the first bad step, and exchanging

the two resulting parts. This correspondence is implied in Figure 2; see [10] for complete details.

Since each $|\mathcal{B}_i| = |\mathcal{A}| = \binom{a+b-1}{a}$, we see that the number of good paths is

$$\binom{a+b}{a} - (k+1)\binom{a+b-1}{a} = \frac{a-kb}{a+b}\binom{a+b}{a}. \quad \square$$

CONCLUSION. It is curious that (1) André’s original proof has generally been forgotten (or at least absent from all current literature), (2) André has been erroneously credited with the reflection method, and finally, (3) while the reflection method does not provide a means for solving the generalized ballot problem, André’s original method does.

We direct the reader to [10] for a translation of André’s proof, details on the modification of his proof to the generalized ballot problem, and more history surrounding the proofs of the ballot theorem. Furthermore, the interested reader should see [11] for four proofs of the generalized ballot theorem (not including the one presented in this paper). The excellent paper [12] also covers the history surrounding the ballot problem, and explores various generalizations.

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