An Introduction to Curvature in Differential Geometry with Some Discussion on Problems of Prescribing Curvature

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Abstract:

Given a smooth connected subset of Euclidean space with codimension one, that is a hypersurface (such as a curve in $\mathbb{R}^2$ or a surface in $\mathbb{R}^3$), we would like to examine geometric measures curvature, which are measures of how non-flat the hypersurface is. Some recently solved and open research problems related to prescribing curvature shall also be discussed.

The Length of a Curve in $\mathbb{R}^2$:

Given a real-valued function of a real variable, $y = f(x)$, consider the graph of this function over some subset $[a, b]$ of the domain of $f(x)$.

The first quantity that we wish to discuss is the so-called **length** L of the curve given by the graph of $y = f(x)$ over $[a, b]$. For simplicity, let us assume* that $f(x)$ is in $C^1([a, b])$, that is, $y = f(x)$ is differentiable on $[a, b]$, with a continuous first derivative on $[a, b]$.

*Otherwise, even with an assumption of continuity, the function may not be of bounded variation, and thus its length may be infinite. For example, consider $y = f(x)$ on $[0, 1]$ given by $f(0) = 0$ and $f(x) = x \sin(\frac{1}{x})$ for $0 < x \leq 1$.

We will assume that the length of a line segment is the only length thus known, and this is simply the value you obtain using the distance formula with the endpoints of the line segment.

So consider a partition of the interval $[a, b]$

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

and define

$$\triangle x_i := x_i - x_{i-1}.$$

For $1 \leq i \leq n$, consider the graph of $y = f(x)$ on $[x_{i-1}, x_i]$. Assuming that $\triangle x_i$ is sufficiently small, by the differentiability of $f(x)$, we have that the length of the graph of $f(x)$ on $[x_{i-1}, x_i]$ is approximately

$$l_i := \sqrt{\triangle x_i^2 + (f(x_i) - f(x_{i-1}))^2}.$$

This is simply the length of the line segment with endpoints $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$.

However, we may rewrite l_i as

$$l_i = \sqrt{1 + (\frac{f(x_i) - f(x_{i-1})}{\triangle x_i})^2} \triangle x_i.$$

Moreover, using the mean value theorem, we see that there is a number $\xi_i \in (x_{i-1}, x_i)$ so that

$$l_i = \sqrt{1 + (f'(\xi_i))^2} \triangle x_i.$$

Thus, L may be approximated by

$$\sum_{i=1}^{n} \sqrt{1 + (f'(\xi_i))^2} \triangle x_i$$

which is a Riemann sum for the integral

$$\int_a^b \sqrt{1 + (f'(x))^2} dx.$$

Moreover, by our regularity assumptions on f, the larger n gets in the above approximation for L, the smaller

$$|L - \sum_{i=1}^{n} \sqrt{1 + (f'(\xi_i))^2} \triangle x_i|$$

gets. Thus

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

Now, consider a parameterized curve in $\mathbb{R}^2$:

$$\vec{r}(t) = (x(t), y(t)),$$

$t \in [a, b]$ and $x(t), y(t)$ in $C^1([a, b])$.*

For parameterized curves $\vec{r}(t)$, we wish to determine the **length** L of the curve over the interval $[a, b]$.

*If $x(t) = t$, then this case reduces to the graph of a function $y = f(t)$, which we just discussed.

Once again, we partition $[a, b]$ by

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b,$$

and define $\triangle t_i = t_i - t_{i-1}$.

Let l_i be the length of the piece of $\vec{r}(t)$ over $[t_{i-1}, t_i]$. Then we may approximate l_i by

$$\sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2}$$

which by the mean value theorem, may be written as

$$\sqrt{(x'(\xi_i))^2 + (y'(\eta_i))^2} \triangle t_i$$

for some $\xi_i, \eta_i \in (t_{i-1}, t_i)$.

Hence, we see that

$$L \approx \sum_{i=1}^{n} \sqrt{x'(\xi_i))^2 + (y'(\eta_i))^2} \triangle t_i$$

for n sufficiently large.

Letting $n \to \infty$ we get

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt.$$

Using the notation $\frac{d\vec{r}}{dt}(t) = (x'(t), y'(t))$ and $|\frac{d\vec{r}}{dt}(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$ we see that

$$L = \int_a^b |\frac{d\vec{r}}{dt}(t)| dt.$$

If we consider a general vector-valued function of one variable

$$\vec{r}(t) = (x_1(t), x_2(t), \cdots, x_m(t)),$$

then using a similar notation to above, we may see that in the case where each of the $x_i(t)'s$ are in $C^1([a,b])$, the length of the curve given by $\vec{r}(t)$, on $[a,b]$ is given by

$$L = \int_a^b |\frac{d\vec{r}}{dt}(t)| dt.$$

Arclength as a Parameter for a Curve:

Consider a vector-valued function $\vec{r}(t)$ as discussed above. If $\vec{r}$ is in $C^1([a,b],\mathbb{R}^n)$ *, then the following function

$$s(t) := \int_a^t |\frac{d\vec{r}}{d\tau}(\tau)|d\tau$$

is the length of $\vec{r}$ on the interval $[a,t]$.

If we assume $|\frac{d\vec{r}}{dt}| \neq 0$ as well, then by the Fundamental Theorem of Calculus we see that

$$s'(t) = |\frac{d\vec{r}}{d\tau}(t)| > 0.$$

Thus we may parameterize $\vec{r}$ by using the so-called **arc length** parameter s.

*meaning each of its component functions are in $C^1([a,b])$

The Curvature of Parameterized Curve in $\mathbb{R}^2$:

Let $\vec{r}(t) = (x(t), y(t))$ be a parameterized curve in $\mathbb{R}^2$, for $t \in [a, b]$. The tangent line to this curve at the point $\vec{r}(t)$ has direction vector $\frac{d\vec{r}}{dt}(t) = (x'(t), y'(t))$. Let θ be the angle between the vector $\frac{d\vec{r}}{dt}$ and the vector $(1, 0)$. Thus

$$\cos\theta = \frac{x'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}}.$$

The **curvature** of the curve given by $\vec{r}(t)$ at any point $\vec{r}(t_0)$ is a measure of how quickly the curve $\vec{r}(t)$ is pulling away from the tangent line

$$(x(t_0) + tx'(t_0), y(t_0) + ty'(t_0))$$

to $\vec{r}(t)$ at t_0.

In order to define this quantity, let us assume that $\vec{r}$ is twice differentiable with respect to its parameter variable.

If we wish to define curvature in a way that only depends on the geometry of the curve $\vec{r}(t)$ at $\vec{r}(t_0)$, we need to make sure we define it in a way which is independent of the parameterization of the curve.

We define the **curvature** of $\vec{r}(t)$ at $\vec{r}(t_0)$, denoted by $\kappa(\vec{r}(t_0))$, to be the quantity

$$\kappa(\vec{r}(t_0)) := \frac{d\theta}{ds}|_{\vec{r}(t_0)}.^*$$

*Where here s is the arc length parameter.

By the chain rule, we see

$$\frac{d}{dt}\cos\theta = -\sin\theta\frac{d\theta}{dt}$$

where $\sin\theta = \frac{y'(t)}{\sqrt{(x'(t))^2+(y'(t))^2}}$ and

$$\frac{d\theta}{ds} = \frac{d\theta}{dt}\frac{dt}{ds}$$

$$= \frac{d\theta}{dt}\frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}}.$$

Moreover,

$$\frac{d}{dt}\cos\theta = \frac{d}{dt}\frac{x'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}}$$

$$= \frac{x''(t)(y'(t))^2 - y''(t)x'(t)y'(t)}{[(x'(t))^2 + (y'(t))^2]^{\frac{3}{2}}}.$$

Thus, we see that

$$\kappa = \frac{x'(t)y''(t) - y'(t)x''(t)}{[(x'(t))^2 + (y'(t))^2]^{\frac{3}{2}}}.$$

In the case where $\vec{r}(t) = (t, f(t))$, this reduces to

$$\kappa = \frac{f''(t)}{[1 + (f'(t))^2]^{\frac{3}{2}}}.$$

Note that if the curve is a graph of a function $y = f(x)$, its curvature $\kappa = \frac{f''(x)}{[1+(f'(x))^2]^{\frac{3}{2}}} = 0$ if and only if $f''(x) = 0$, and thus if and only if $f(x)$ is a linear function.

In the case where the curve is given parametrically by $\vec{r}(t)$, it turns out that $\kappa = 0$ if and only if the curve is a line. To see this, we examine another way in which we can calculate the curvature of the curve.

Recall that we define θ to be the angle between $(x'(t), y'(t))$ and $(1, 0)$ at the point $(x(t), y(t))$.

Moreover, if we parameterize the curve by arc length s, we have $|\frac{d\vec{r}}{ds}| = 1$, and thus $\frac{d\vec{r}}{ds} = (\cos\theta, \sin\theta)$.

Differentiating with respect to s, and using that $\kappa = \frac{d\theta}{ds}$, we see that

$$\frac{d^2\vec{r}}{ds^2} = \kappa(-\sin\theta, \cos\theta)$$

and thus

$$|\frac{d^2\vec{r}}{ds^2}| = |\kappa|.$$

Hence, $\kappa \equiv 0$ if and only if $x''(s), y''(s) \equiv 0$, and this is if and only if $\vec{r} = \vec{a}s + \vec{b}$.

A Variational Approach to Finding the Curvature:

Recall that the length of a curve L which is represented as the graph of a function $y = f(x)$ on $[a, b]$ is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

Let

$$f_\varepsilon(x) = f(x) + \varepsilon g(x)$$

where $g(a) = g(b) = 0$. Let us define L_ε to be the length of the graph of f_ε on $[a, b]$.

Then the first variation of L is given by

$$\frac{dL_\varepsilon}{d\varepsilon}|_{\varepsilon=0} = \frac{d}{d\varepsilon}|_{\varepsilon=0}\int_a^b \sqrt{1+(f'(x)+\varepsilon g'(x))^2}dx$$

$$= \int_a^b \frac{\partial}{\partial\varepsilon}|_{\varepsilon=0}\sqrt{1+(f'(x)+\varepsilon g'(x))^2}dx$$

$$= \int_a^b \frac{f'(x)}{\sqrt{1+(f'(x))^2}}g'(x)dx$$

$$= -\int_a^b \frac{f''(x)}{[1+(f'(x))^2]^{\frac{3}{2}}}g(x)dx.$$

So critical values of the functional

$$L(f) = \int_a^b \sqrt{1+(f'(x))^2}dx$$

are functions f whose curvature κ vanishes everywhere, i.e. straight lines.

Some Examples of Curves, and Their Corresponding Curvature Functions:

1. Consider a parabolic function

$$f(x) = ax^2 + bx + c.$$

The curvature κ at $(x, f(x))$ on the graph is given by

$$\kappa = \frac{2a}{[1 + (2ax + b)^2]^{\frac{3}{2}}}.$$

Note that $|\kappa|$ is largest at the vertex of the parabola, and $\kappa \to 0$ as $x \to \pm\infty$.

2. $\vec{r}(t) = (R\cos t, R\sin t)$, $t \in [0, 2\pi]$, is a circle of radius R centered at the origin. A calculation shows

$$\kappa = \frac{1}{R}$$

at all points on the circle.*

*It can be shown that if $\mathcal{C}$ is a curve with constant curvature $k > 0$ at each point on $\mathcal{C}$, then $\mathcal{C}$ is an arc of a circle of radius $\frac{1}{k}$.

3. Suppose that a curve is expressed parametrically by polar coordinates, that is, as a graph over the unit circle. Then

$$\vec{r}(\phi) = (\rho(\phi)\cos\phi, \rho(\phi)\sin\phi),$$

for $\phi \in [0, 2\pi]$. The curvature of this curve is given by

$$\kappa = \frac{-\rho''\rho + 2(\rho')^2 + \rho^2}{[\rho^2 + (\rho')^2]^{\frac{3}{2}}}.$$

If we assume that the curve does not contain the origin, then $u(\phi) = \frac{1}{\rho(\phi)}$ exists for all ϕ. In such a case, the curvature at the point $(\rho(\phi)\cos\phi, \rho(\phi)\sin\phi)$ satisfies

$$\frac{u^3}{[u^2 + (u')^2]^{\frac{3}{2}}}(u'' + u) = \kappa.$$

This leads us to consider the question: If we were to prescribe κ as a function of ϕ, or even as a function defined in the entire plane $\mathbb{R}^2$, what are the solutions to the above non-linear second order differential equation?

If we were to solve it for u, we would have a curve which is the graph over $\mathbb{S}^1$, the unit circle, whose curvature is given by κ. This is essentially the **Minkowski** and **Weingarten curvature problems** in the plane.

The Minkowski* Problem:

Given a function $k : [0, 2\pi] \to \mathbb{R}$, we may ask the question: When is k the curvature of a curve $\vec{r}(t)$ whose unit normal is $-(\cos t, \sin t)$? If $k > 0$, then the resulting curve, if it exists, will be convex. Moreover, this problem can be solved in the case where $k > 0$ if and only if

$$\int_0^{2\pi} \frac{\cos\theta}{k(\theta)} d\theta = \int_0^{2\pi} \frac{\sin\theta}{k(\theta)} d\theta = 0.$$

*Actually a one dimensional equivalent to the so-called Minkowski Problem.

The Weingarten Curvature Problem:

Give a function $k : \mathbb{R}^2 \to \mathbb{R}$, we may ask the question: Does there exist a curve $\mathcal{C}$ whose curvature κ is given by the function $k|_{\mathcal{C}}$? That is, in the case where $\mathcal{C}$ is given parametrically by $\vec{r}(t)$, we require $\kappa(\vec{r}(t)) = k(\vec{r}(t)) \; \forall t$.*

*We will address this question in higher dimensions later in the discussion as well.

Curvature of curves in $\mathbb{R}^3$:

Suppose $\vec{r}$ is a parameterization of a smooth curve by arc length. We may define the curvature of the curve at the point $\vec{r}(s)$ to be the quantity

$$\kappa = |\frac{d^2\vec{r}}{ds^2}(s)|.$$

It turns out that $\frac{d^2\vec{r}}{ds^2} \perp \frac{d\vec{r}}{ds}$. To see this, note that $\frac{d\vec{r}}{ds} \cdot \frac{d\vec{r}}{ds} = 1$, and differentiating this, we get $2\frac{d\vec{r}}{ds} \cdot \frac{d^2\vec{r}}{ds^2} = 0$. Thus

$$\frac{d^2\vec{r}}{ds^2} = \kappa\vec{n}$$

where $\vec{n}$ is a unit vector perpendicular to $\vec{t} := \frac{d\vec{r}}{ds}$. Defining $\vec{b} := \vec{t} \times \vec{n}$, we have that $\vec{t}, \vec{n}, \vec{b}$ is a basis for $\mathbb{R}^3$ at each value of s.

The Frenet Formulas:

$$\frac{d\vec{t}}{ds} = \kappa\vec{n},$$

$$\frac{d\vec{n}}{ds} = -\kappa\vec{t} - \tau\vec{b},$$

$$\frac{d\vec{b}}{ds} = \tau\vec{n}.$$

Here κ is the **curvature** of the curve, which measures how the curve is bending, and τ is the so-called **torsion**, which measures how the curve is twisting.

Curvature for Surfaces S in $\mathbb{R}^2$:

Let $\mathcal{C}$ be a curve on S, a surface in $\mathbb{R}^2$, and let $p \in \mathcal{C}$. Let κ be the curvature of $\mathcal{C}$ at p (with $\vec{n}$ as defined earlier), and suppose we may define a differentiable function $N : S \to \mathbb{S}^2$, where $N(p)$ is perpendicular to S at p.

Let $\vec{n} \cdot N = \cos\theta$. Then we define the **normal curvature of $\mathcal{C}$ at p** by

$$\kappa_n = \kappa \cos\theta.$$

It turns out that there is a theorem of *Meusnier* that says that all curves lying on S having at a given point the same tangent line, have at that point the same normal curvature.

The maximum and minimum values of the normal curvatures at $p \in S$ are defined to be the so-called **principal curvatures** of S at p. It turns out that these are the eigenvalues for the linear transformation $-dN_p$. To see this, consider $-dN_p(\alpha'(0)) \cdot \alpha'(0)$ where $\alpha(s) \in S$ with $\alpha(0) = p$ and $' = \frac{d}{ds}$. Then

$$-dN_p(\alpha'(0)) \cdot \alpha'(0) = N(\alpha(0)) \cdot \alpha''(0)$$

$$= N \cdot \kappa \vec{n}|_p = \kappa_n(p).$$

Curvature of Hypersurfaces in Higher Dimensional Euclidean Space:

- Let $M \subset \mathbb{R}^{n+1}$ be a smooth, orientable, regular hypersurface, i.e. M is locally a solution to an equation of the type*

$$G(x_1, \cdots, x_{n+1}) = 0,$$

  with

$$\nabla G(x_1, \cdots, x_{n+1}) \neq 0.$$

- M is **orientable** if $\exists$ differentiable $N : M \to \mathbb{S}^n = \{\vec{x} \in \mathbb{R}^{n+1} : \|\vec{x}\| = 1\}$. N is called the **Gauss Map**.

*This condition guarantees, via the implicit function theorem, that we may write M locally as a graph of a function on n variables.

The Gauss Map:

We examine how the Gauss map N varies at a point $p \in M$.

- $dN_p : T_pM \to T_{N(p)}\mathbb{S}^n \equiv T_pM$
  - T_pM is the tangent space to M at p.
  - $T_{N(p)}\mathbb{S}^n$ is the tangent space to $\mathbb{S}^n$ at $N(p)$.
  - Meaning of notation: $\alpha(t) \in M$ a curve with $\alpha(0) = p$, $\alpha'(0) \in T_pM$, $n(t) := N(\alpha(t))$ and $dN_p(\alpha'(0)) = n'(0)$.

More Background Information:

- The eigenvalues $\kappa_1(p), \cdots, \kappa_n(p)$ of $-dN_p$ are called the **Principal Curvatures** of M at p, which are real since $-dN_p$ is self-adjoint.

- $\Pi_p(\cdot,\cdot) = < -dN_p(\cdot), \cdot >$: $T_pM \times T_pM \to \mathbb{R}$ is called the **Second Fundamental Form**.

  – If $\{e_1, \cdots, e_n\}$ is an orthonormal basis for T_pM, then the eigenvalues of $\Pi_p(e_i, e_j)$ are the principal curvatures.

- If M is expressed locally as a graph $(\vec{x}, u(\vec{x}))$ around $p = 0$, with $u(0) = 0$, $\nabla u(0) = 0$, then the eigenvalues of $D^2u(0)$ are the principal curvatures of M at p.

Symmetric Functions of the Principal Curvatures:

- **Mean Curvature** of M at p:

$$H(p) = \frac{1}{n}(\kappa_1(p) + \cdots + \kappa_n(p))$$

- **Gauss-Kronecker Curvature** of M at p:

$$K(p) = \kappa_1(p) \cdots \kappa_n(p)$$

Some Problems of Prescribing Curvature – The Weingarten Curvature Problem:

- Given $F : \mathbb{R}^{n+1} \to \mathbb{R}$, what hypersurfaces M satisfy

  $$H(\vec{X}) = F(\vec{X}),$$

  or

  $$K(\vec{X}) = F(\vec{X})$$

  $\forall \vec{X} \in M$?

- To see the complexity of this question, we first ask: What M satisfy $H \equiv 0$?
  - $M \subset \mathbb{R}^3$ with $H \equiv 0$ is called a **minimal surface**.

Minimal Surfaces:

- Minimal surfaces are critical points for the functional $A[M] = \int_M dA$.
  - If the surface M is given by $f(t_1, t_2)$, for u small, define a *normal variation* $\alpha(u, t_1, t_2) = f(t_1, t_2) + u\varphi(t_1, t_2)\vec{N}(t_1, t_2)$. Let $M(u)$ be the surface defined by $\alpha(u, \cdot, \cdot)$. Then

$$\frac{d}{du}A[M(u)]\bigg|_{u=0} = -\int_M 2\varphi H dA.$$

Examples:

- **catenoid**: Only non-planar surface of revolution minimal surface (rotate $z = \cosh x$ about the x-axis).

- **helicoid**: Only non-planar ruled minimal surface.

The Plateau Problem:

- Of the surfaces with a given boundary, which have the smallest possible area?
  - This you can solve experimentally by dipping your curve, as realized by a piece of wire, in soapy water.

- Problem was posed by the Belgian Physicist Plateau in 1850.

- For a fixed Jordan curve $C \subset \mathbb{R}^3$, in 1930 Douglas and Radó showed $\exists S$ s.t. $\partial S = C$ and S disk-like having minimum area.

Prescribing Gauss-Kronecker Curvature:

Consider the problem of **prescribing Gauss-Kronecker curvature**, with $F > 0$.

- By a theorem of **Hadamard**, any hypersurface M with $K(\vec{X}) = F(\vec{X})$, $\forall \vec{X} \in M$, must be convex.

- Case $F \equiv 1$: $K_{\mathbb{S}^n} \equiv 1$. It can be shown that $\mathbb{S}^n$ is the only solution. For uniqueness results of this type, see:
  - Aeppli (Proc. Amer. Math. Soc. '60)
  - A.D. Alexandrov (AMS Transl. '62)
  - Ros and Korevaar (JDG '88)
  - Li (Cont. Math. '97)

The PDE Connection:

To find M such that $K(\vec{X}) = F(\vec{X})\ \forall \vec{X} \in M$, we use tools of **fully non-linear second order elliptic partial differential equations**.

The Case in $\mathbb{R}^2$:

- For a curve $C \subset \mathbb{R}^2$ given in polar coordinates by $(x, y) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$ the curvature of C is given by

$$k = \frac{-r_{\theta\theta} r + 2r_\theta^2 + r^2}{(r_\theta^2 + r^2)^{\frac{3}{2}}}.$$

- Given $F : \mathbb{R}^2 \to \mathbb{R}$, can we find C so that $F|_C = k$?

- Writing $u = \frac{1}{r}$ the differential equation for u is

$$\frac{u^3}{(u_\theta^2 + u^2)^{\frac{3}{2}}}(u_{\theta\theta} + u) = F(\frac{1}{u}(\cos\theta, \sin\theta)).$$

Details:

- We look for M star-shaped of the form

$$M = \{\rho(\vec{x})\vec{x} : \vec{x} \in \mathbb{S}^n\}.$$

- Then $u = \rho^{-1}$ must satisfy:

$$(u^2 + |\nabla u|^2)^{-\frac{n+2}{2}} u^{n+2} \frac{\det(\nabla_{ij} u + u e_{ij})}{\det(e_{ij})} = F(\frac{1}{u(\vec{x})}\vec{x}),$$

  - $e_{ij} =< e_i, e_j >$ is the usual metric for $\mathbb{S}^n$
  - $e_1, \cdots, e_n$ is a local frame
  - ∇ is covariant differentiation on $\mathbb{S}^n$

- This equation for u is of **Monge-Ampère** type.

THEOREM (Oliker Comm. PDE '84) 1 *Let F be a smooth positive function defined on the set $\{\vec{X} \in \mathbb{R}^{n+1} : r_1 \leq |\vec{X}| \leq r_2\}$, $r_1 < 1 < r_2$*

$$F(\vec{X})|\vec{X}|^n \geq 1 \qquad \forall |\vec{X}| = r_1, \tag{1}$$

$$F(\vec{X})|\vec{X}|^n \leq 1 \qquad \forall |\vec{X}| = r_2, \tag{2}$$

and

$$\frac{\partial}{\partial \rho}(\rho^n F(\rho \vec{x})) \leq 0 \tag{3}$$

for each fixed $\vec{x} \in \mathbb{S}^n$. Then there exists a smooth hypersurface M with $K(\vec{X}) = F(\vec{X})$ $\forall \vec{X} \in M$. Moreover, any two solutions are endpoints of a one parameter family of homothetic dilations, all of which are solutions to $F(\vec{X}) = K(\vec{X})$.

The Method of Continuity:

- In solving an equation of the type

$$H(\vec{x}, u, \nabla u, \nabla^2 u) = 0,$$

  we need that the linearized operator

$$Lw = H_{\nabla^2 u} \nabla^2 w + H_{\nabla u} \nabla w + H_u w$$

  at a solution u, is invertible.

  - For this, we only need to check that the null space of L is 0.

- Oliker needed $\frac{\partial}{\partial \rho}(\rho^n F(\rho \vec{x})) \leq 0$ to:

  - conclude that the null space is zero;

  - obtain the uniqueness up to homothety.

Further Results:

- Delanoë (Ann. Sci. Ecole Norm Sup. '85) was able to drop this monotone condition at the expense of uniqueness.

- Caffarelli, Nirenberg and Spruck (Current Topics in PDE's '86): Prescribing more general symmetric functions of the principal curvatures.

- Bakelman and Kantor (Geom. and Top., Leningrad '74), Treibergs and Wei (JDG '83): Prescribing mean curvature.

The Topological Degree Theory Approach:

Tso (JDG '91) raised the question: Prescribing K when F bounded between two positive constants.

- Oliker's result does not cover this case.

- An alternate approach to this PDE, is via **degree theory** argument.

  - The advantage is you do not need to know invertibility of the linearized operator.

  - Require that F is invariant under a fixed-point free $G \subset O(n+1)$.

- Li (Cont. Math. '97) showed: F bounded and G-invariant $\Rightarrow \exists M$ with Gauss-Kronecker curvature F.

- Li showed: F strictly monotone in one direction $\Rightarrow$ there doesn't exist M with Gauss-Kronecker curvature $F|_M$.
  - Example: $F(\vec{X}) = \tan^{-1}(X_{n+1}) + 10\pi$.

THEOREM (Mikula 2006) 1 *Let $F \in C^{2+\alpha}(\mathbb{R}^3 \setminus \{0\})$, $\alpha \in (0,1)$, G-invariant, where $G \subset O(3)$ is fixed point free. Suppose F also satisfies*

$$\limsup_{|\vec{X}| \to 0} F(\vec{X})|\vec{X}|^2 < 1,$$

$$\liminf_{|\vec{X}| \to \infty} F(\vec{X})|\vec{X}|^2 > 1.$$

Then there exists G-invariant $M := \{\rho(\vec{x})\vec{x} : \vec{x} \in \mathbb{S}^2\}$ with Gauss-Kronecker curvature $K(\vec{X}) = F(\vec{X})\ \forall \vec{X} \in M$.

- Growth conditions on F are optimal: For $\varepsilon > 0$, the functions $F(\vec{X}) = (1 \pm \varepsilon)|\vec{X}|^{-2}$ do not admit M with $K(\vec{X}) = F(\vec{X})\ \forall \vec{X} \in M$.

An Open Problem:

Open Problem: 1 *Let* $F \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$*, G-invariant, $G \subset O(n+1)$ fixed-point free. Suppose F also satisfies*

$$\limsup_{|\vec{X}|\to 0} F(\vec{X})|\vec{X}|^n < 1,$$

$$\liminf_{|\vec{X}|\to \infty} F(\vec{X})|\vec{X}|^n > 1.$$

Show that there exists a G-invariant hypersurface $M := \{\rho(\vec{x})\vec{x} : \vec{x} \in \mathbb{S}^n\}$ with Gauss-Kronecker curvature $K(\vec{X}) = F(\vec{X})\ \forall \vec{X} \in M$.

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