Using Analog Circuits to Motivate the Laplace Transform in Introductory Differential Equations Courses.

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Abstract.

An overview of how standard Laplace transform techniques can be used in solving analog circuit problems, providing instructors of introductory differential equations courses with lively examples for the purpose of illustration.

Immediately after covering the standard Laplace transform methods for solving initial value problems with discontinuous forcing functions, I introduce the three passive analog circuit elements: resistors, inductors, and capacitors. I define resistors to be devices which dissipate electrical energy in the form of heat and whose governing equation

$$v_R(t) = Ri(t) \tag{1}$$

is known as Ohm's Law. Similarly, inductors are defined to be devices in which electrical energy is stored in the form of a magnetic field and for which the governing ODE is

$$v_L(t) = Li'(t) \quad . \tag{2}$$

Finally, capacitors are defined to be devices in which electrical energy is stored in the form of an electric field and for which the governing equation is

$$i_c(t) = \frac{C\,dv_c}{dt} \qquad \text{or equivalently,} \qquad v_c(t) = \frac{\int_0^t i(\tau)\,d\tau}{c} \quad . \tag{3}$$

Taking the Laplace transforms of (1), (2), and (3), we obtain

$$\mathbf{V}_{\mathrm{R}}(s) = R\,\mathbf{I}(s) \quad , \tag{4}$$

$$\mathbf{V}_{\mathrm{L}}(s) = Ls\,\mathbf{I}(s) \quad , \tag{5}$$

and

$$\mathbf{V}_{\mathrm{C}}(s) = 1/(Cs)\ \mathbf{I}(s) \quad , \tag{6}$$

where the well known properties

$$\mathbf{L}\{f'(t)\} = s\,\mathbf{L}\,\{f(t)\} - f(0)$$

and

$$L\left\{\int_0^t i(\tau)\,\mathrm{d}\tau\right\} = \frac{L\,\{f(t)\}}{s}$$

have been used. Results (4), (5), and (6) could be viewed as generalizations of Ohm's Law to the *s* domain with *R*, *Ls*, and 1/(*Cs*) serving as the resistive, inductive, and capacitative impedances to the flow of current. Hence, RLC circuits composed of series arrangemants of an inductor, resistor, and capacitorpowered by a voltage source *v*(*t*) could be represented in the time domain through the integro-differential equation

$$L\,i'(t) + R\,i(t) + 1/C\int_0^t i(\tau)\ d\tau\,\mathrm{d}\tau = v(t)$$

or equivalently in the *s* domain through the algebraic equation

$$Ls\mathbf{I}(s) + R\mathbf{I}(s) + 1/(Cs)\,\mathbf{I}(s) = \mathbf{V}(s)$$

where **I**(*s*) and **V**(*s*) denoted the Laplace transforms of *i*(*t*) and *v*(*t*), respectively.

Applying Kirchoff's Voltage Law (KVL), we recognize that the portion of the overall voltage drop accounted for by the capacitor is

$$H(s) = \frac{L\,s}{L\,s + R + \dfrac{1}{C\,s}} \quad ,$$

or, equivalently

$$H(s) = \frac{L\,s^2}{L\,s^2 + R\,s + \frac{1}{C}} \qquad ,$$

where $\mathbf{H}(s)$, defined to be the ratio $\mathbf{Y}(s)$ / $\mathbf{X}(s)$, denotes the transfer function relating the *s*-domain characteristics of the output signal to those of the input signal. It is clear that our configuration of the RLC circuit acts as a high pass filter (HPF), given that

$$\mathbf{H}(0) = 0 \qquad \text{and} \qquad \lim_{|s| \to \infty} H(s) = 1 \quad ,$$

so that the circuit severely attenuates low frequency signals but effectively multiplies higher frequency ones by one.

Alternate arrangements of the resistor, capacitor, and inductor yield low pass filters (LPF's), which pass low frequency signals but severely attenuate higher frequency ones, as well as band pass filters (BPF's), which severely attenuate low and high frequency signals but pass those of an intermediate range of frequencies. Such filters can be useful in detecting and estimating signals whose frequencies differ markedly from those of ambient noise signals. The values of *R*, *L*, and *C* can be adjusted to produce the desired filter characteristics.

More generally, the function of analog filters can be summarized through the equation

$$\mathbf{Y}(s) = \mathbf{H}(s)\,\mathbf{X}(s) \qquad , \qquad (7)$$

where the transfer function $\mathbf{H}(s)$ is the weighing factor relating the output signal to the input signal in the *s* domain.

As a direct consequence of the well known convolution theorem for Laplace transforms, we observe that (7) can be reformulated in the time domain in a manner expressing the output signal $y(t)$ as the convolution of $h(t)$ with the input signal $x(t)$

$$y(t) = \int_0^t h(t-\tau)\, x(\tau)\, d\tau \quad ,$$

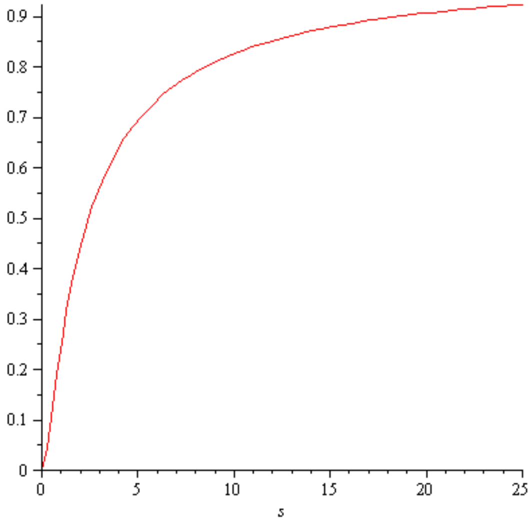
where $h(t)$ denotes the inverse Laplace transform of the transfer function $\mathbf{H}(s)$. For linear, time-invariant (LTI) systems, it is well known that $h(t)$ represents such a system's response to the Dirac delta distribution, and is known as the filter's *impulse response*.

Operational Amplifiers

If one limits oneself to passive circuit elements in implementing LPF, HPF, and BPF filters, the resulting performance may be disappointing. For example, in a hypothetical HPF filter implemented through RLC circuits, we might obtain a transfer function of the form

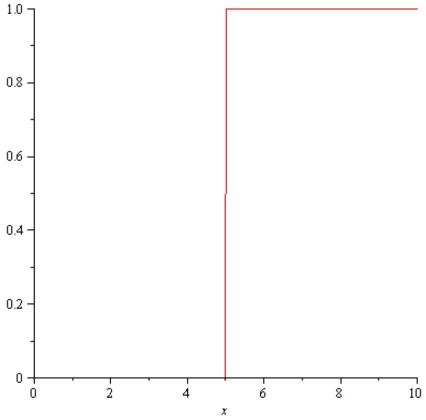
$$\mathbf{H}(s) = \frac{s^2}{s^2 + 2 \cdot s + 1} \quad .$$

It is evident from the graph of $\mathbf{H}(s)$



that for the transitional frequency range $3 < s < 12$, there is ambiguity as to whether incoming signals are being attenuated or are being multiplied by one.

The ideal HPF characteristics would entail an instantaneous transition from zero to one at a specified frequency, as is illustrated below



.

It can be shown ([5]) that these ideal filter characteristics could never be obtained, as such a filter would be non causal, requiring the present value of the output signal $y(t)$ to be determined by more than just the present and past values of the input signal $x(t)$.

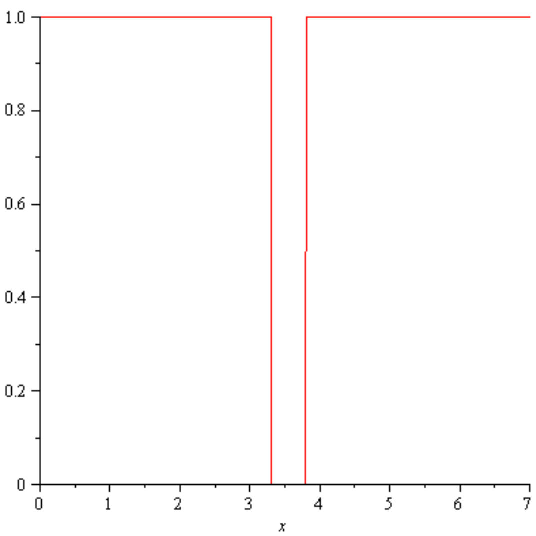
Nevertheless, significant improvements can be obtained by incorporating operational amplifiers (op amps), devices constructed from transistors, diodes, resistors, and capacitors and capable of magnifying input voltages by a factor of 10^5. Op amps are modeled through singular perturbation problems of the form

$$\varepsilon v'_o(t) = \psi(v_i(t)),$$

where the parameter ε is on the order of 10^{-5}, the function ψ approximates the Heaviside operator, and $v_0(t)$ and $v_i(t)$ denote the output and input voltages, respectively.

Through the use of resistors, capacitors, and op amps, HPF transfer functions can be produced with much steeper ascents and a much narrower range of transitional frequencies. Similar improvements can be obtained in the performance of LPF and BPF filters.

Arrangements of resistors, capacitors, and op amps also allow for the implementation of notch filters, whose ideal frequency domain characteristics are shown below:



Notch filters are critical components in signal extraction from noise problems in which most of the ambient noise's frequencies are concentrated over a very narrow range. It is evident from (7) that for such circumstances, the application of notch filters to incoming signals embedded in noise allows for an almost perfect recovery of the original signal along with a suppression of most of the noise. Notch filters are also used to protect electrical equipment from resonance effects; the filters can be designed to suppress all signals of frequencies close to the equipment's resonant frequencies.

Conclusion

A question that instructors of ODE courses are likely to be asked by students is

> "*Now that the world has gone digital and that even analog signals are transmitted after time sampling, of what relevance are analog circuits and the analysis of signals and systems of continuous time ? What are the benefits of covering these topics besides being given a chance to apply Laplace transform techniques ?*"

Such skepticism may be allayed by reminding students of the need to establish an interface between digital circuitry and the predominantly analog world of day to day experience.

Illustrations such as the ones found in ([1]), p. 156 may prove to be even more convincing. These circuit diagrams depict commonly used models for JFET and MOSFET transistors, in which these devices are depicted as arrangements of capacitors, resistors, and dependent current sources.

Evidently, the inner workings of transistors can only be understood if these are modeled as analog circuits. Given that transistors are essential components of NAND gates, XOR gates, flip flops, and virtually all other digital circuitry gates, the advent of the digital age has in no way diminished the need to study analog processes.

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